Theory of Hypothesis Test II

Data Driven Healthcare

Mod B

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Tests of the Mean of a Normal Population Sigma Unknown (1 of 2)

- Convert sample result $(\bar{x})$ to a *t* test statistic

Hypothesis Tests for μ

σ Known

σ Unknown

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

Reject H_0 if $t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}} > t_{n-1,\, \alpha}$

Tests of the Mean of a Normal Population sigma unknown (2 of 2)

- For a two-tailed test:

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1,\frac{\alpha}{2}} \text{ or if } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\frac{\alpha}{2}}$$

Example 7: Two-Tail Test Sigma Unknown

The average cost of a vaccine dose is said to be $168. A random sample of 25 hospitals resulted in

$\overline{x} = \$172.50$ and

$s = \$15.40$. Test at the

$\alpha = 0.05$ level.

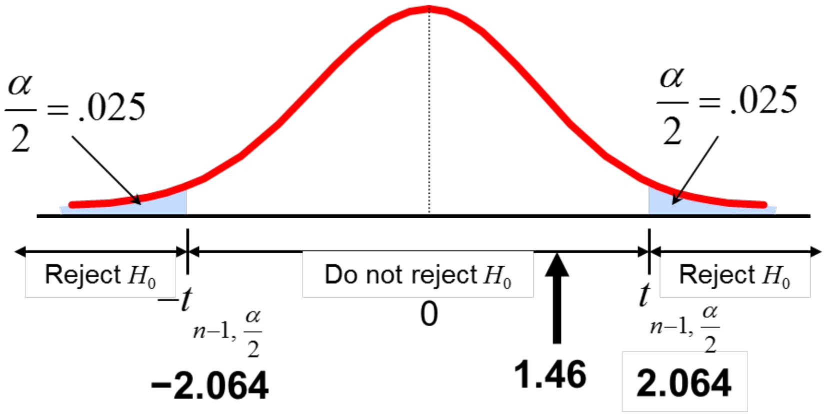
$$H_0 : \mu = 168$$

$$H_1 : \mu \neq 168$$

Example Solution: Two-Tail Test

$H_0: \mu = 168$

$H_1: \mu \neq 168$



- $\alpha = 0.05$
- $n = 25$
- σ is unknown, so use a *t* statistic

$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

- Critical Value:

$$t_{24, .025} = \pm 2.064$$

Do not reject H_0: not sufficient evidence that true mean cost is different than $168

Tests of the Population Proportion (1 of 2)

- Involves categorical variables
- Two possible outcomes
  - "Success" (a certain characteristic is present)
  - "Failure" (the characteristic is not present)
- Fraction or proportion of the population in the "success" category is denoted by P
- Assume sample size is large

Tests of the Population Proportion (2 of 2)

- The sample proportion in the success category is denoted by $\hat{p}$

$$\hat{p} = \frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$$

- When $nP(1-P) > 5,\ \hat{p}$ can be approximated by a normal distribution with mean and standard deviation

$$\mu_{\hat{p}} = P \qquad \sigma_{\hat{p}} = \sqrt{\frac{P(1-P)}{n}}$$

Hypothesis Tests for Proportions

- The sampling distribution of $\hat{p}$ is approximately normal, so the test statistic is a z value:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1 - P_0)}{n}}}$$

Hypothesis Tests for *P*

$nP(1 - P) > 5$

$$H_0 : P = P_0$$
$$H_1 : P > P_0$$

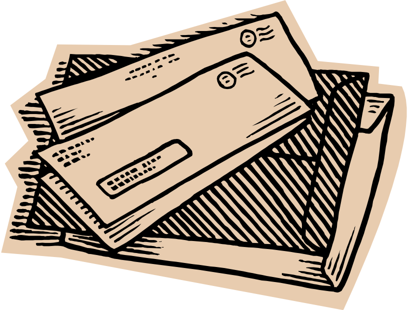
$nP(1 - P) < 5$

Not discussed in this chapter

Example 8: *Z* Test for Proportion

A medical devices company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the

$\alpha = .05$ significance level.



Check:

Our approximation for *P* is

$$\hat{p} = \frac{25}{500} = .05$$

$$nP(1-P) = (500)(.05)(.95)$$
$$= 23.75 > 5 \checkmark$$

Z Test for Proportion: Solution

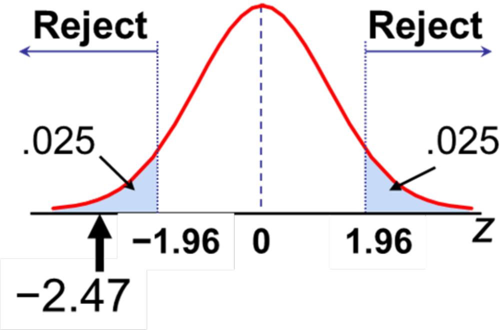
$$H_0: P = .08$$

$$H_1: P \neq .08$$

$$\alpha = .05$$

$$n = 500,\ \hat{p} = .05$$

Critical Values: ± 1.96



Test Statistic:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Decision:

Reject H_0 at $\alpha = .05$

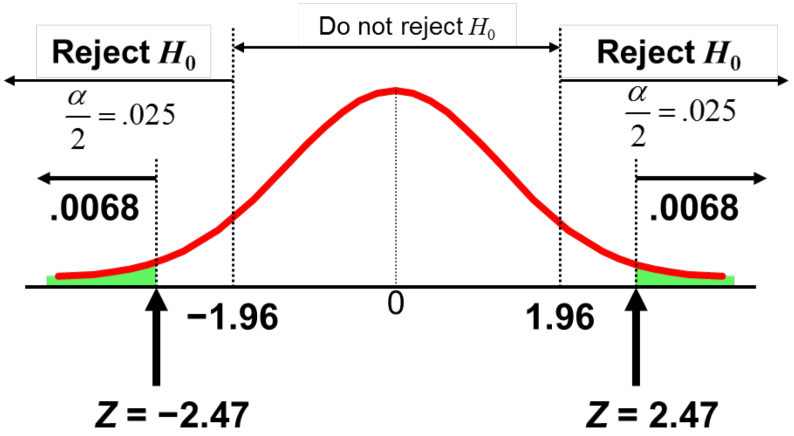
Conclusion:

There is sufficient evidence to reject the company's claim of 8% response rate.

p-Value Solution

Calculate the p-value and compare to α

(For a two sided test the p-value is always two sided)



p-value = .0136:

$$P(Z \le -2.47) + P(Z \ge 2.47)$$
$$= 2(.0068) = 0.0136$$

Reject H_0 **since p-value** $= .0136 < \alpha = .05$

Assessing the Power of a Test

- Recall the possible hypothesis test outcomes:

Key:
Outcome
(Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**Correct Decision** ($1 - \alpha$)	**Type II Error** (β)
Reject H_0	**Type I Error** (α)	**Correct Decision** ($1 - \beta$)

- β denotes the probability of Type II Error
- $1 - \beta$ is defined as the power of the test

Power $= 1 - \beta$ = the probability that a false null hypothesis is rejected

Type II Error

Assume the population is normal and the population variance is known.
Consider the test

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu > \mu_0$$

The decision rule is:

Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > z_\alpha$ or Reject H_0 if $\bar{x} > \bar{x}_c = \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}}$

If the null hypothesis is false and the true mean is μ^*,

then the probability of type II error is

$$\beta = P(\bar{x} < \bar{x}_c \mid \mu = \mu^*) = P\left(z < \frac{\bar{x}_c - \mu^*}{\dfrac{\sigma}{\sqrt{n}}} \right)$$

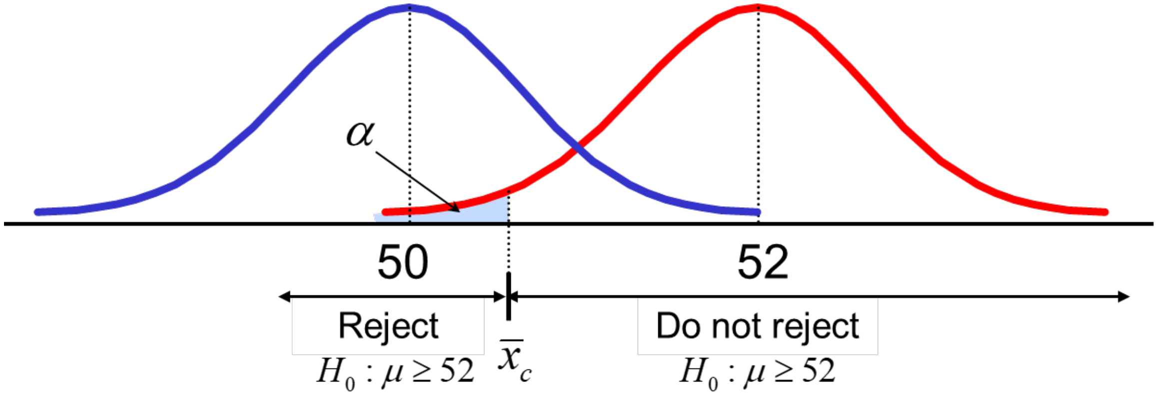
Type II Error Example (1 of 3)

- Type II error is the probability of failing

  to reject a false H_0

  Suppose we fail to reject $H_0 : \mu \geq 52$

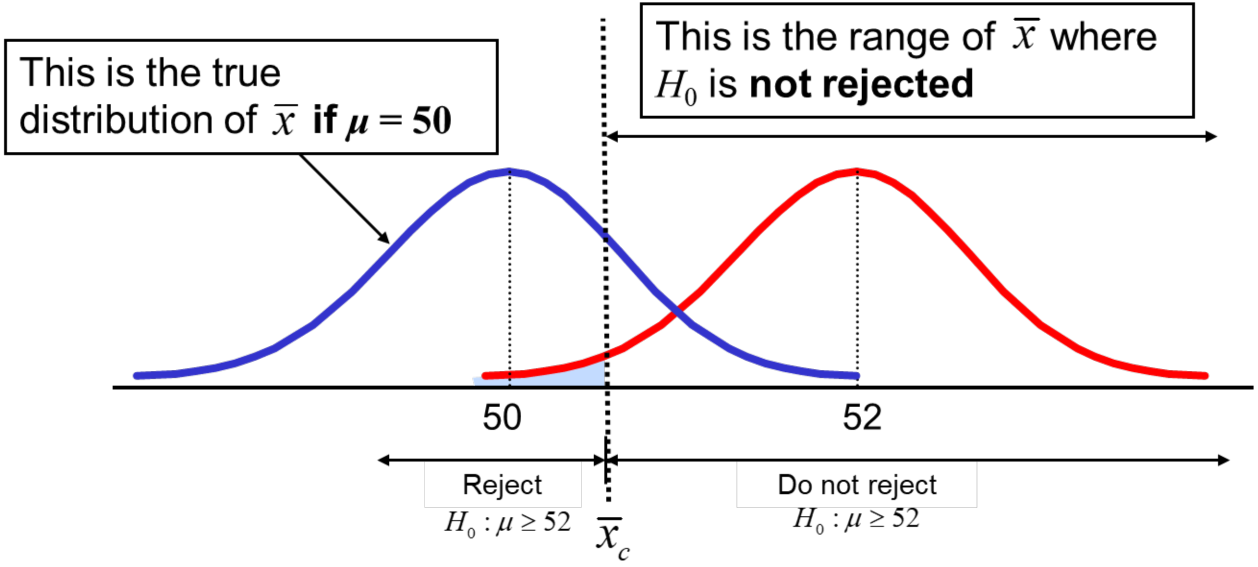
  when in fact the true mean is $\mu^* = 50$



Type II Error Example (2 of 3)

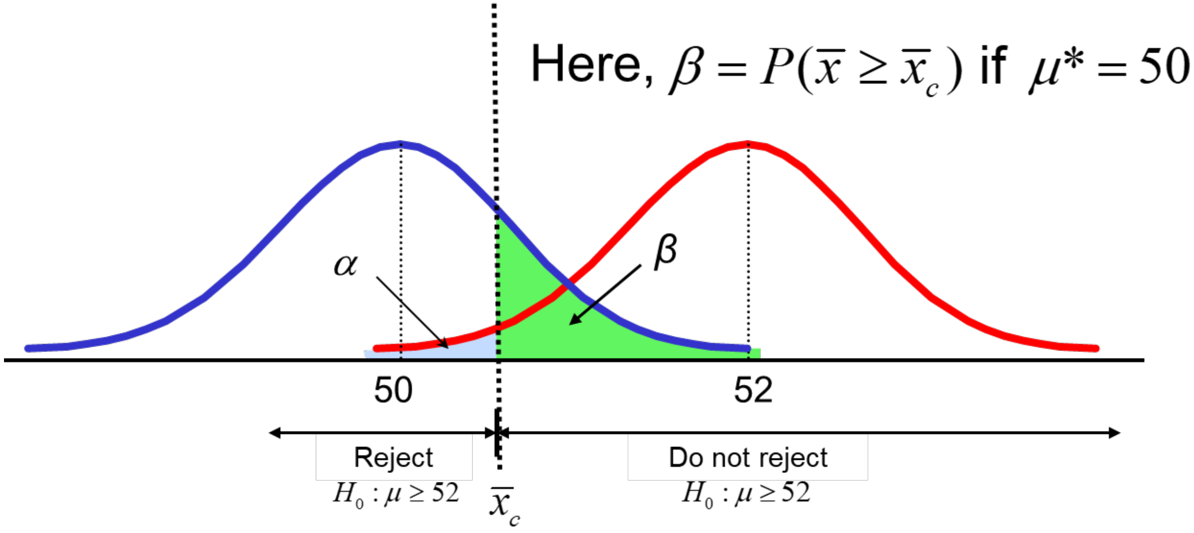
- Suppose we do not reject $H_0: \mu \geq 52$ when in fact

  the true mean is $\mu^* = 50$



Type II Error Example (3 of 3)

- Suppose we do not reject $H_0 : \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



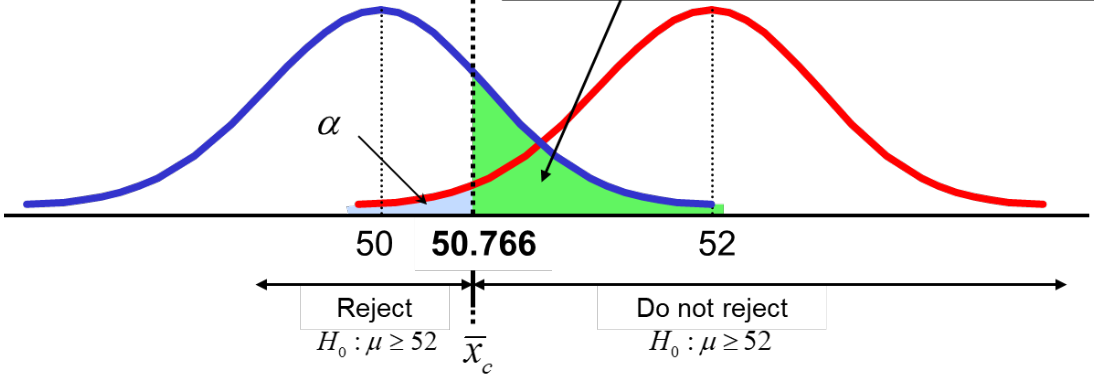
Calculating *Beta* (1 of 2)

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

$$\bar{x}_c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = \boxed{50.766}$$

(for $H_0 : \mu \geq 52$)

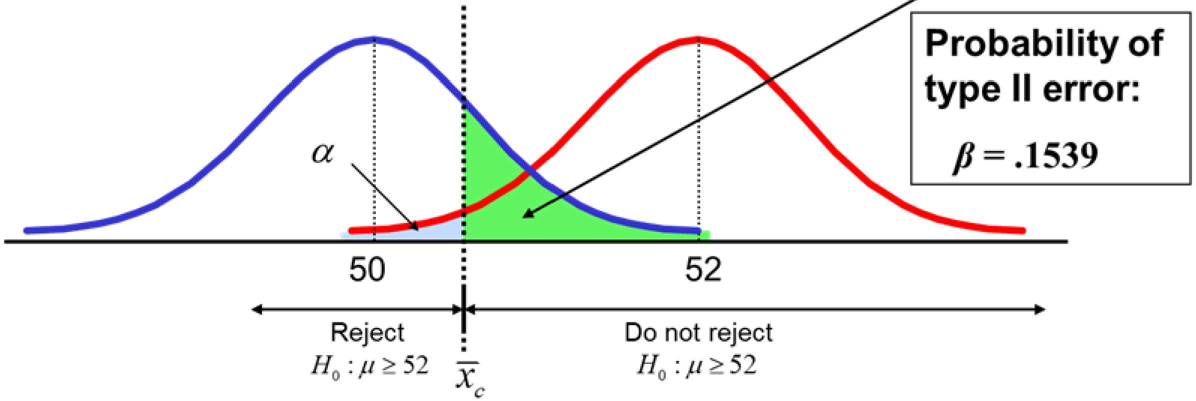
So $\beta = P(\bar{x} \geq 50.766)$ if $\mu^* = 50$



Calculating *Beta* (2 of 2)

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

$$P(\bar{x} \geq 50.766 \mid \mu^* = 50)\ P = \left(z \geq \frac{50.766 - 50}{\frac{6}{\sqrt{64}}} \right) = P(z \geq 1.02) = .5 - .3461 = \boxed{.1539}$$



Power of the Test Example

If the true mean is $\mu^* = 50,$

- The probability of Type II Error $= \beta = 0.1539$
- The power of the test $= 1 - \beta = 1 - 0.1539 = 0.8461$

Key:
Outcome
(Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**Correct Decision** $1 - \alpha = 0.95$	**Type II Error** $\beta = 0.1539$
Reject H_0	**Type I Error** $\alpha = 0.05$	**Correct Decision** $1 - \beta = 0.8461$

(The value of β and the power will be different for each μ^*)

Tests of the Variance of a Normal Distribution (1 of 2)

- Goal: Test hypotheses about the population variance,

  σ^2 (e.g., $H_0 : \sigma^2 = \sigma_0^2$)

  - If the population is normally distributed,

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

has a chi-square distribution with $(n-1)$ degrees of freedom

Tests of the Variance of a Normal Distribution (2 of 2)

The test statistic for hypothesis tests about one population variance is

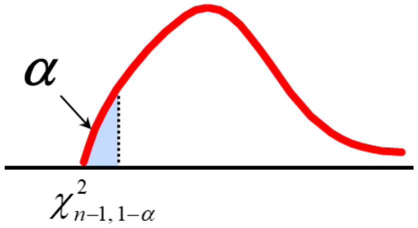
$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

Decision Rules: Variance

Population variance

Lower-tail test:

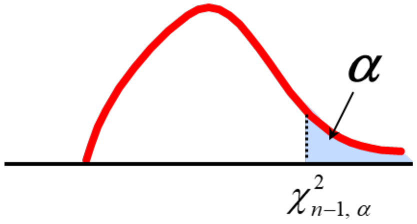
$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$H_1 : \sigma^2 < \sigma_0^2$$



Reject H_0 if
$\chi^2_{n-1} < \chi^2_{n-1, 1-\alpha}$

Upper-tail test:

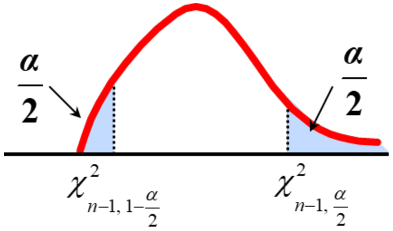
$$H_0 : \sigma^2 \leq \sigma_0^2$$
$$H_1 : \sigma^2 > \sigma_0^2$$



Reject H_0 if
$\chi^2_{n-1} > \chi^2_{n-1, \alpha}$

Two-tail test:

$$H_0 : \sigma^2 = \sigma_0^2$$
$$H_1 : \sigma^2 \neq \sigma_0^2$$



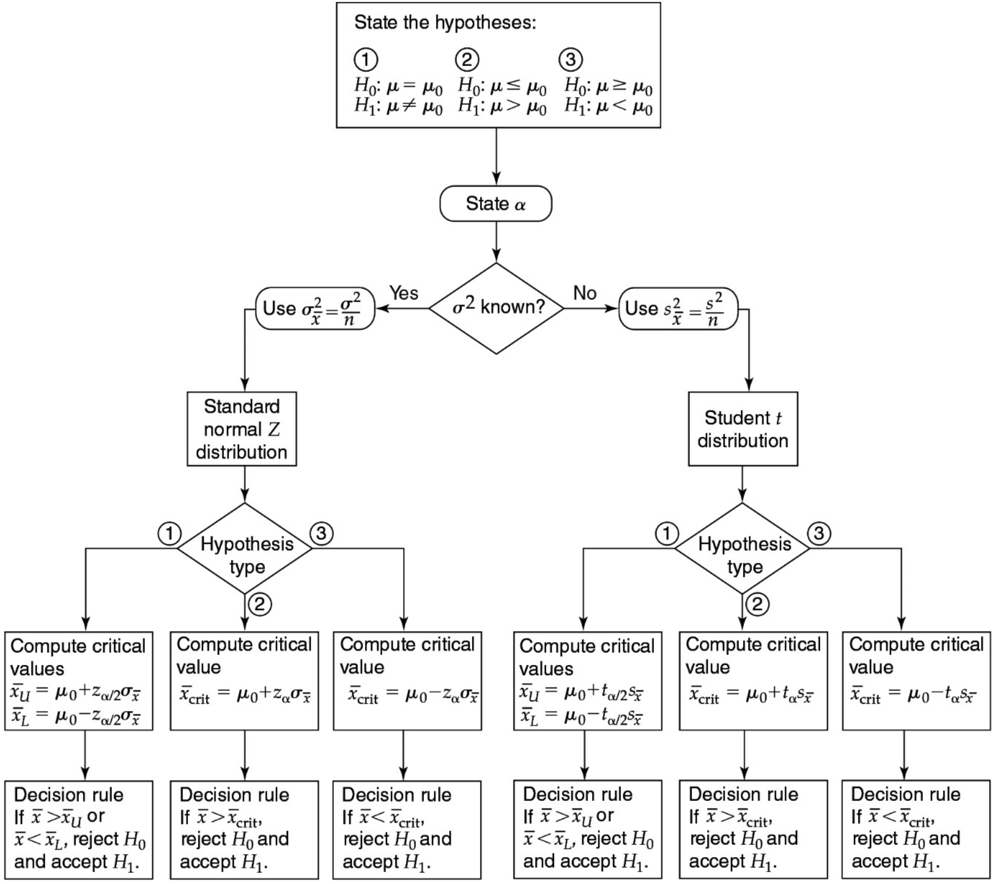
Reject H_0 if
$\chi^2_{n-1} > \chi^2_{n-1, \frac{\alpha}{2}}$
or
$\chi^2_{n-1} < \chi^2_{n-1, 1-\frac{\alpha}{2}}$

Summary

- Addressed hypothesis testing methodology
- Performed *z* Test for the mean $(\sigma \text{ known})$
- Discussed critical value and *p*-value approaches to hypothesis testing
- Performed one-tail and two-tail tests
- Performed *t* test for the mean $(\sigma \text{ unknown})$
- Performed *z* test for the proportion
- Discussed type II error and power of the test
- Performed a hypothesis test for the variance (χ^2)

Appendix: Guidelines for Decision Rule (1 of 2)

Guidelines for Choosing the Appropriate Decision Rule for a Population Mean



Appendix: Guidelines for Decision Rule (2 of 2)

Guidelines for Choosing the Appropriate Decision Rule for a Population Proportion