Theory of Hypothesis Test I

Data Driven Healthcare
Mod B

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Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving
  - a single population mean from a normal distribution
  - a single population proportion (large samples)
  - the variance of a normal distribution
- Formulate a decision rule for testing a hypothesis
- Know how to use the critical value and *p*-value approaches to test the null hypothesis (for both mean and proportion problems)
- Define Type I and Type II errors and assess the power of a test

Concepts of Hypothesis Testing

- A hypothesis is a claim (assumption) about a population parameter:
  - population mean

    Example: The mean expenditure for medical drugs in this city is

    $$\mu = \$52$$
  - population proportion

    Example: The proportion of adults in this city with a specific disease is *P* = .88

The Null Hypothesis, *H* Sub 0 (1 of 2)

- States the assumption (numerical) to be tested

  Example: The average number of Hospital access in a year in Europe is equal to 3.

$$\left(H_0 : \mu = 3\right)$$

- Is always about a population parameter, not about a sample statistic

$$H_0 : \mu = 3 \qquad \cancel{H_0 : \bar{x} = 3}$$

The Null Hypothesis, *H* sub 0 (2 of 2)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- Refers to the status quo
- Always contains "=", "≤" or "≥" sign
- May or may not be rejected

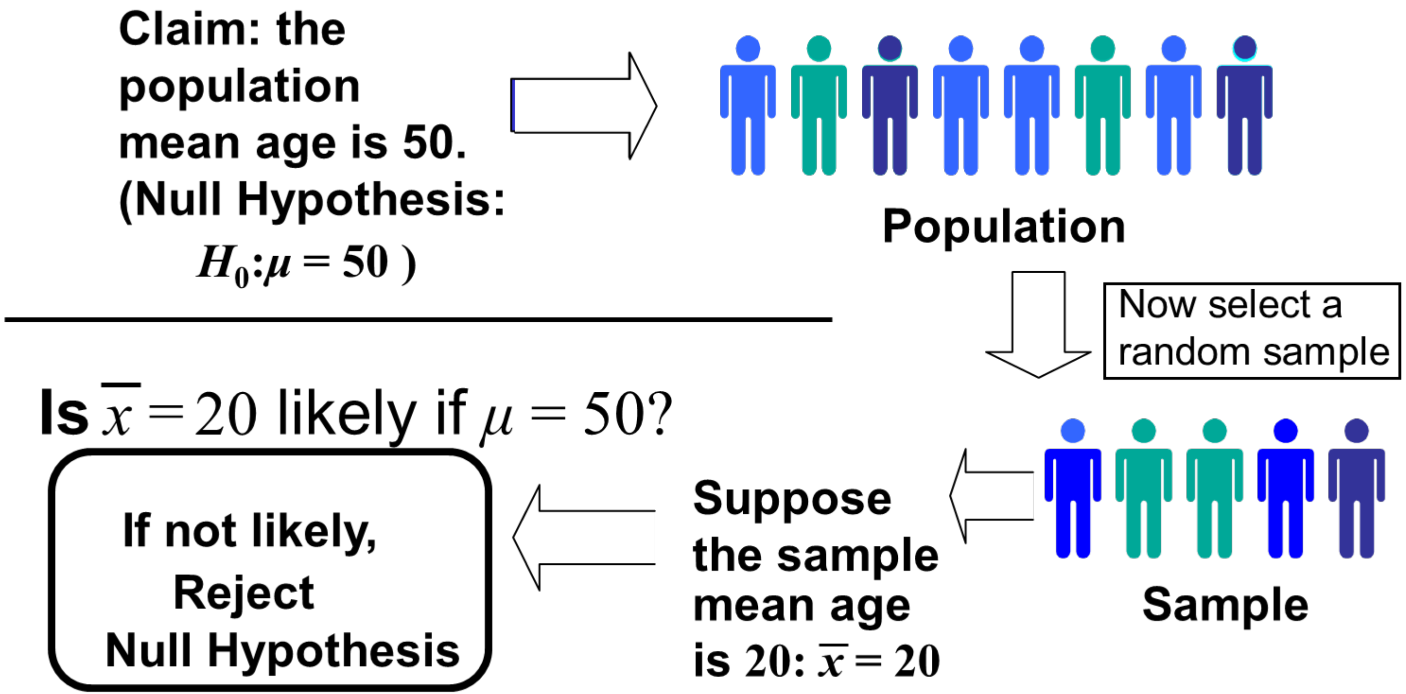
The Alternative Hypothesis, *H* Sub 1

- Is the opposite of the null hypothesis
  - e.g., The average number of Hospital access in a year in Europe is is not equal to 3

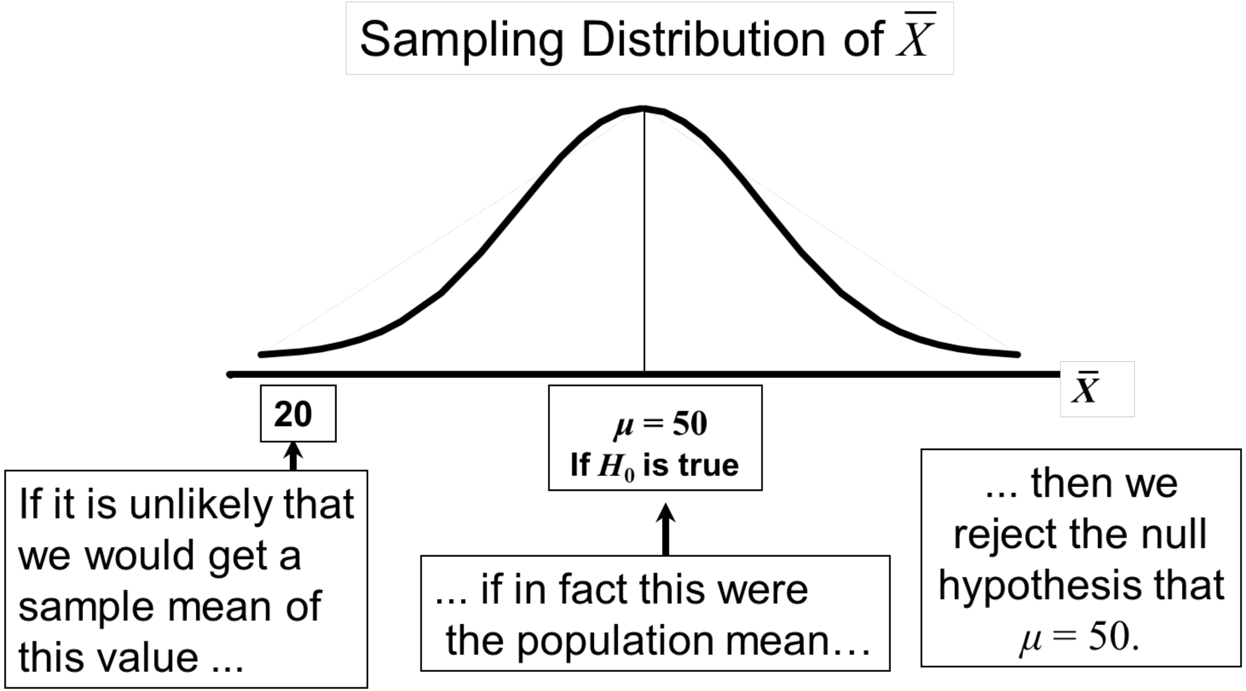
$$(H_1 : \mu \neq 3)$$

- Challenges the status quo
- Never contains the "=", "≤" or "≥" sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

Hypothesis Testing Process



Reason for Rejecting *H* Sub o

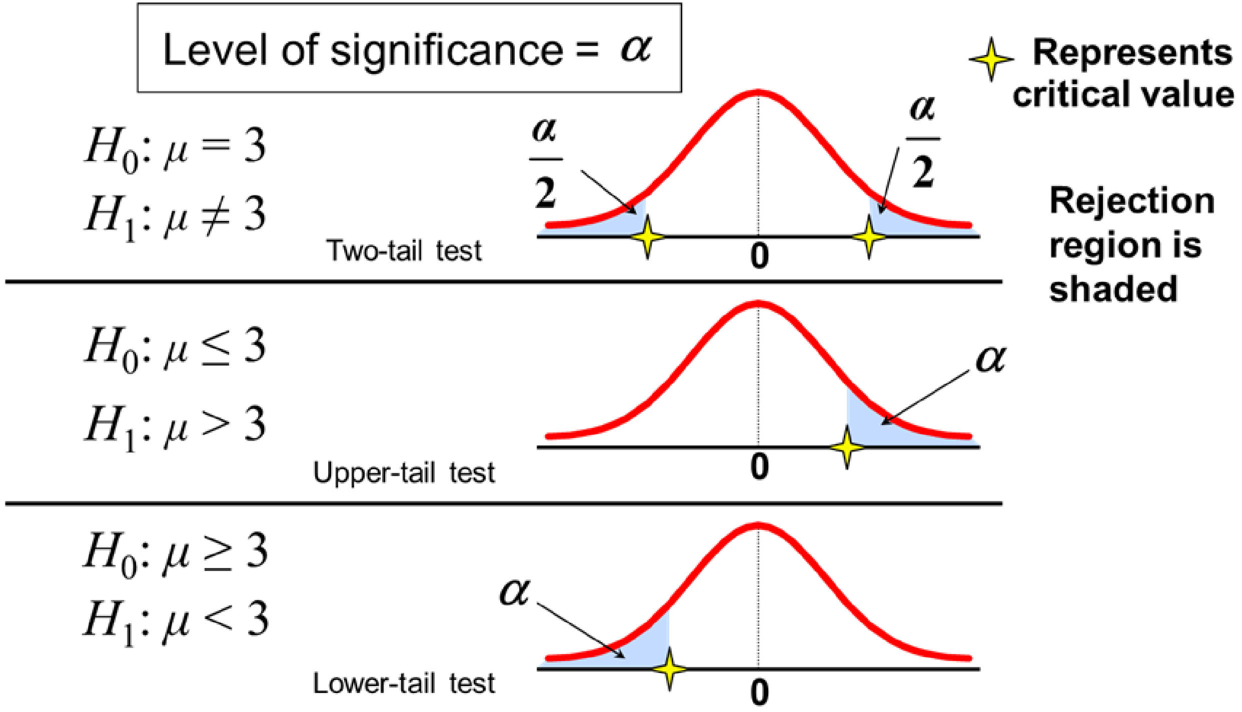


Level of Significance α

- Defines the unlikely values of the sample statistic if the null hypothesis is true
- Defines rejection region of the sampling distribution

- Is designated by **α**, (level of significance)
  - Typical values are 0.01, 0.05, or 0.10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Level of Significance and the Rejection Region



Errors in Making Decisions (1 of 2)

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions (2 of 2)

- **Type II Error**

Fail to reject a false null hypothesis

The probability of Type II Error is β *(beta)*

Outcomes and Probabilities

Possible Hypothesis Test Outcomes

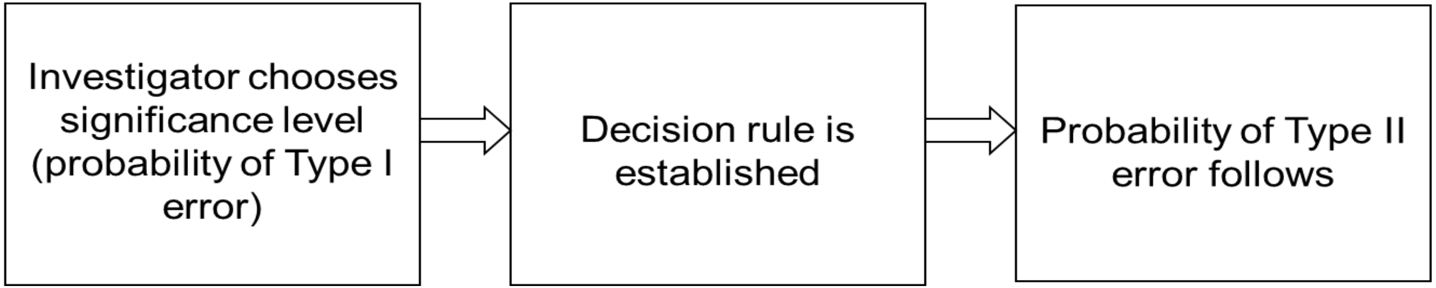
	Actual Situation	
Decision	H_0 True	H_0 False
Fail to Reject H_0	**Correct Decision** $(1-\alpha)$	**Type II Error** (β)
Reject H_0	**Type I Error** (α)	**Correct Decision** $(1-\beta)$

Key:
Outcome
(Probability)

$(1-\beta)$ is called the power of the test

Consequences of Fixing the Significance Level of a Test

- Once the significance level α is chosen (generally less than 0.10), the probability of Type II error, β, can be found.



Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability $(\alpha) \Uparrow$, then

Type II error probability $(\beta) \Downarrow$

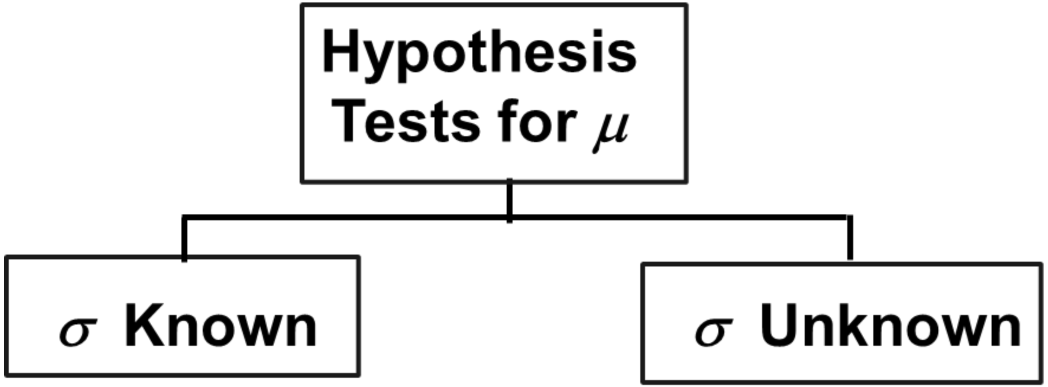
Power of the Test

- The power of a test is the probability of rejecting a null hypothesis that is false

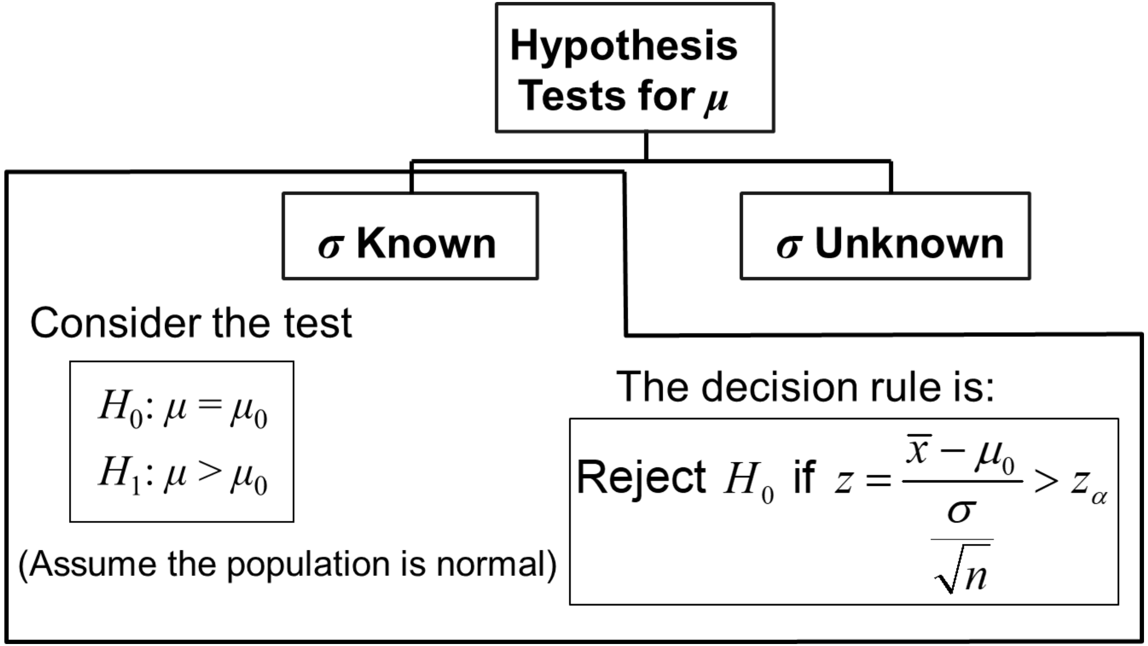
$$\text{Power} = P(\text{Reject } H_0 \mid H_1 \text{ is true})$$

Power of the test increases as the sample size increases

Hypothesis Tests for the Mean



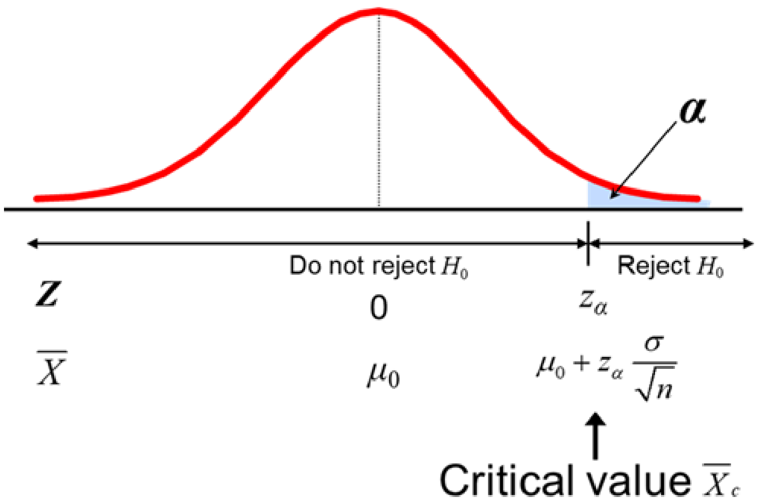
Tests of the Mean of a Normal Distribution Sigma Known



Decision Rule

Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > z_\alpha$

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$



Alternate rule:

Reject H_0 if $\bar{x} > \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}}$

p-Value

- *p*-value: Probability of obtaining a test statistic more extreme

($\leq$ or $\geq$)

than the observed sample value given H_0 is true

Also called observed level of significance

Smallest value of ***α*** for which H_0 can be rejected

p-Value Approach to Testing

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., z statistic)

- Obtain the p-value
  - For an upper tail test:

$$p\text{-value} = P\left(z > \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}, \text{ given thet } H_0 \text{ is true}\right)$$

$$= P\left(z > \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \middle| \mu = \mu_0\right)$$

- Decision rule: compare the p-value to $\boldsymbol{\alpha}$
  - If p-value $< \boldsymbol{\alpha}$, reject H_0
  - If p-value $\geq \boldsymbol{\alpha}$, do not reject H_0

Upper-Tail *Z* Test for Mean Sigma Known

A Hospital thinks that the average age of incoming patients has increased, and now average over 52 years. The hospital wishes to test this claim.

(Assume $\sigma = 10$ is known)

Form hypothesis test:

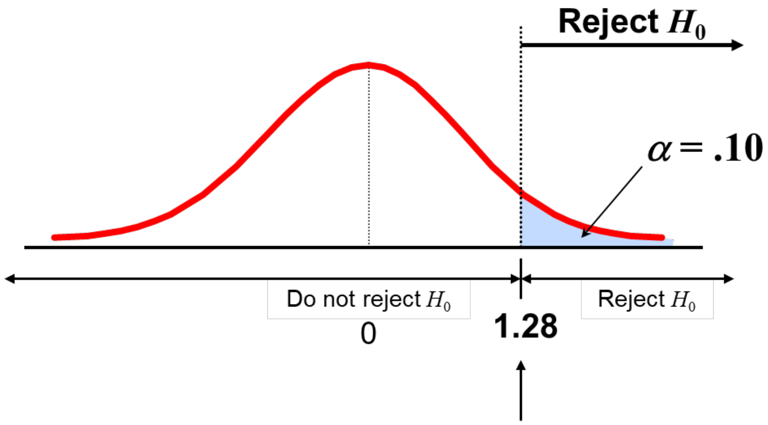
$H_0 : \mu \le 52$ the average is not over 52 years

$H_1 : \mu > 52$ the average is greater than 52 years
(i.e., sufficient evidence exists to support the hospital's claim)

Find Rejection Region

- Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:



Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > 1.28$

Sample Results

Obtain sample and compute the test statistic

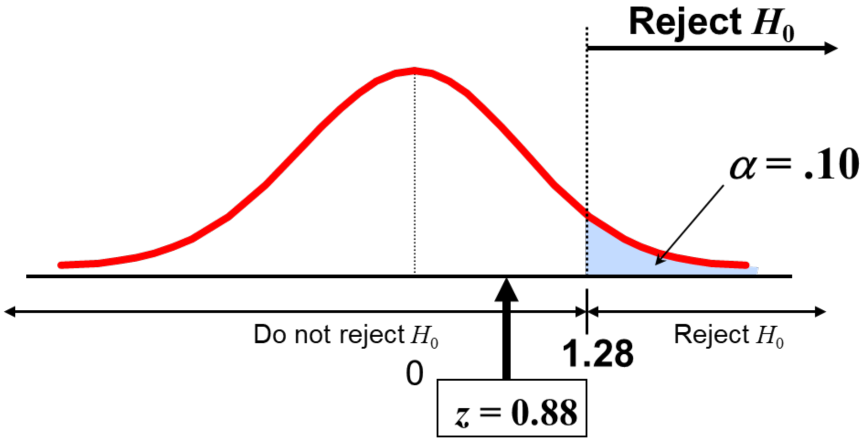
Suppose a sample is taken with the following results: $n = 64,\ \bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Decision

Reach a decision and interpret the result:



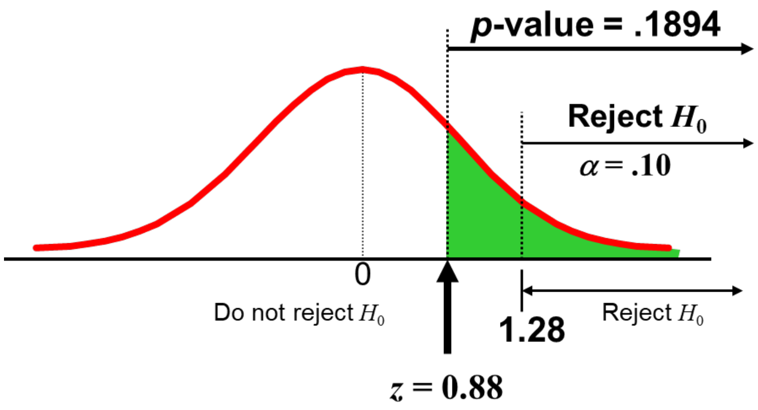
Do not reject H_0 since $z = 0.88 < 1.28$

i.e.: there is not sufficient evidence that the mean average is over 52 years

p-Value Solution

Calculate the p-value and compare to α

(assuming that $\mu = 52.0$)



$$P(\bar{x} \geq 53.1 \mid \mu = 52.0)$$

$$= P\left(z \geq \frac{53.1 - 52.0}{\frac{10}{\sqrt{64}}} \right)$$

$$= P(z \geq 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 **since** p**-value** $= .1894 > \alpha = .10$

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

$$H_0: \mu \leq 3$$
$$H_1: \mu > 3$$

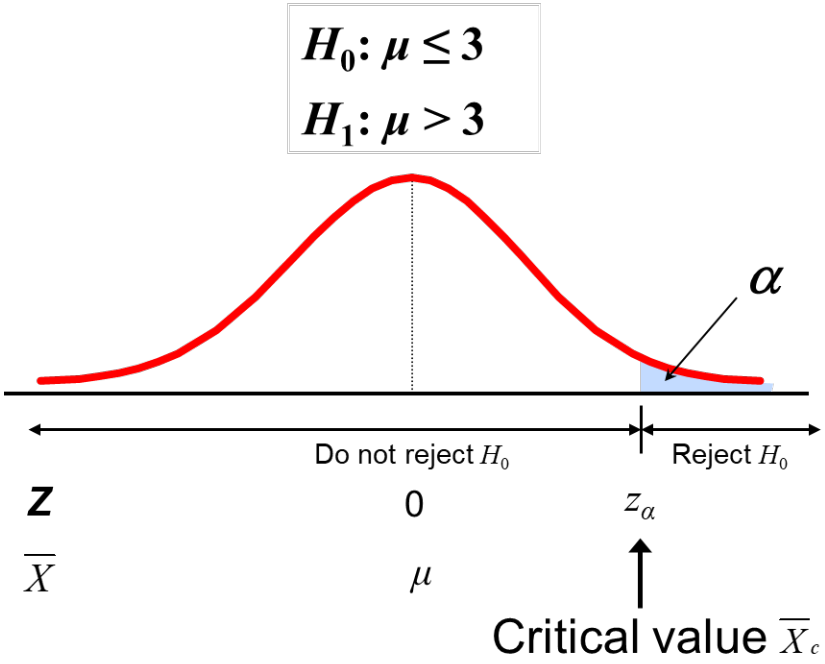
⇨ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

$$H_0: \mu \geq 3$$
$$H_1: \mu < 3$$

⇨ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

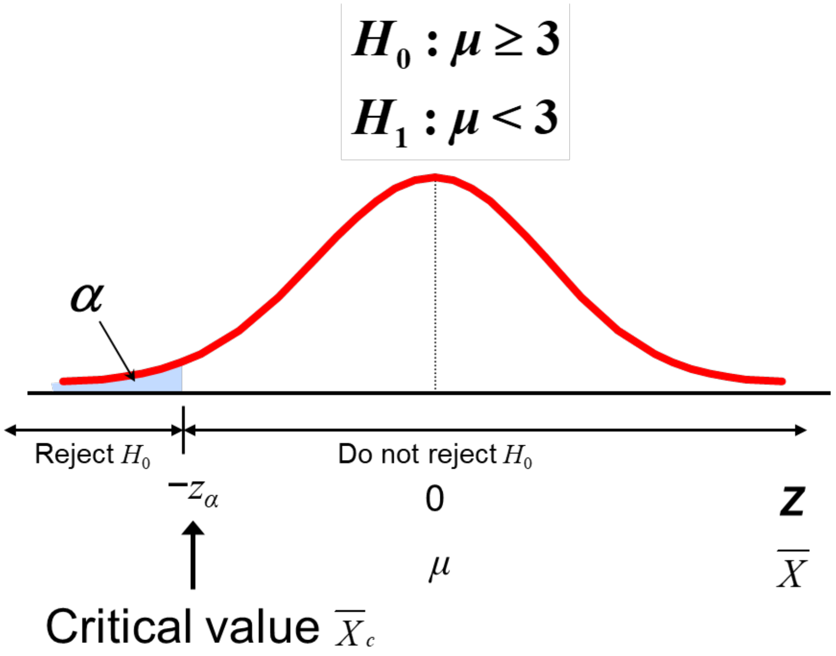
Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

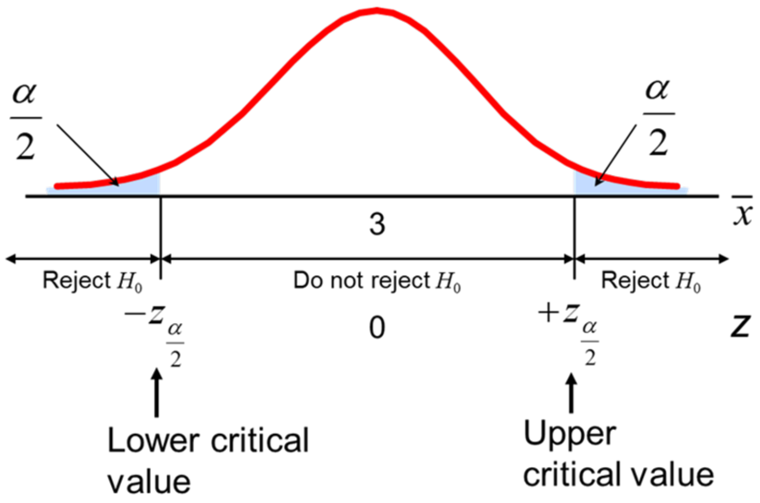


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

- There are two critical values, defining the two regions of rejection

$$H_0 : \mu = 3$$
$$H_1 : \mu \neq 3$$



Hypothesis Testing Example (1 of 4)

Test the claim that the true mean # of access to the hospital is equal to 3

(Assume $\sigma = 0.8$)

- State the appropriate null and alternative hypotheses

  $H_0 : \mu = 3, \ H_1 : \mu \neq 3$ (This is a two tailed test)

- Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test

- Choose a sample size
  - Suppose a sample of size $n = 100$ is selected

Hypothesis Testing Example (2 of 4)

- Determine the appropriate technique

  σ is known so this is a z test

- Set up the critical values
  - For $\alpha = .05$ the critical z values are ± 1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

$$n = 100,\ \bar{x} = 2.84\ (\sigma = 0.8 \text{ is assumed known})$$

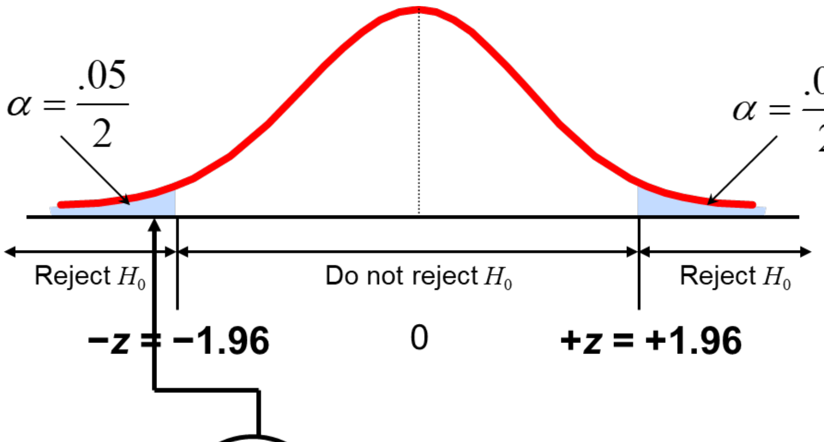
So the test statistic is:

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing Example (3 of 4)

- Is the test statistic in the rejection region?

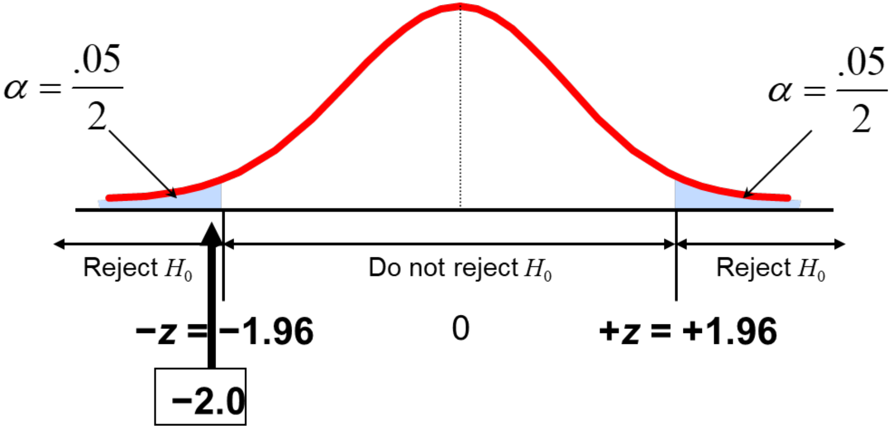
Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0



Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example (4 of 4)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96,$

we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of access to hospital is not equal to 3

p-Value (1 of 2)

- **Example:** How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction)

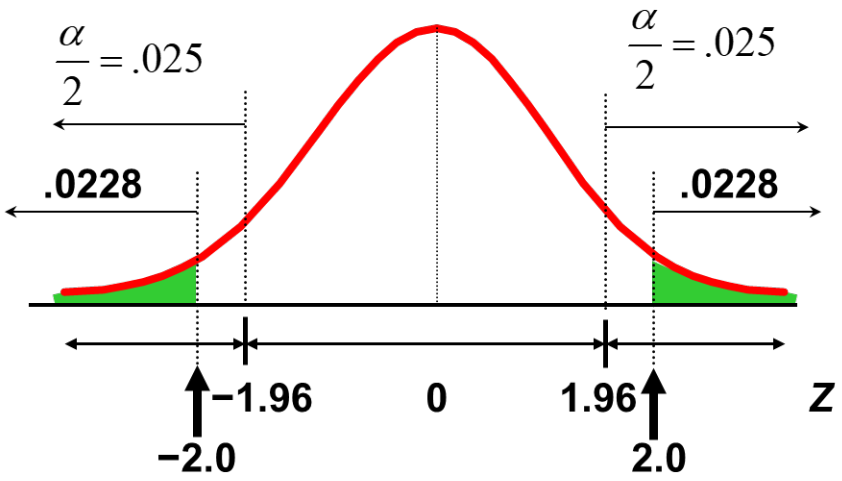
if the true mean is $\mu = 3.0$?

$\bar{x} = 2.84$ is translated to a z score of $z = -2.0$

$$P(z < -2.0) = .0228$$

$$P(z > 2.0) = .0228$$

***p*-value** $= .0228 + .0228 = .0456$



p-Value (2 of 2)

- Compare the p-value to $\boldsymbol{\alpha}$
  - $p\text{-value} < \alpha$, reject H_0
  - $p\text{-value} \geq \alpha$, do not reject H_0

Here: $p\text{-value} = .0456$

$\alpha = .05$

Since .0456 < .05, we reject the null hypothesis